

Percolation of partially interdependent networks under targeted attack

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The study of interdependent networks, and in particular the robustness on networks, has attracted considerable attention. Recent studies mainly assume that the dependence is fully interdependent. However, targeted attack for partially interdependent networks simultaneously has the characteristics of generality in real world. In this letter, the comprehensive percolation of generalized framework of partially interdependent networks under targeted attack is analyzed. As $\alpha = 0$ and $\alpha = 1$, the percolation law is presented. Especially, when $a = b = k$, $p_1 = p_2 = p$, $q_A = q_B = q$, the first and second lines of phase transition coincide with each other. The corresponding phase transition diagram and the critical line between the first and the second phase transition are found. We find that the tendency of critical line is monotone decreasing with parameter p_1 . However, for different α , the tendency of critical line is monotone increasing with α . In a larger sense, our findings have potential application for designing networks with strong robustness and can regulate the robustness of some current networks.

PACS numbers: 89.75.Hc, 64.60.ah, 89.75.Fb

I. INTRODUCTION

Complex networks have been shown to exist in many different areas in the real world and have been intensively studied in recent years. However, almost all network research have been focused on properties of a single network component that does not interact and depend on other networks [1–13]. Such situations rarely, if ever, occur in reality [14–17]. In 2010, Buldyrev et al. [15] developed a theoretical framework for studying the process of cascading failures in one-to-one correspondent interdependent networks caused by random initial failure of nodes. Surprisingly, they found that a broader degree distribution increased the vulnerability of interdependent networks to random failure, which is in contrast to the behavior of a single network. Partially interdependent networks were investigated by Parshani et al. [16], they presented a theoretical framework for studying this case in the same year. Their findings highlighted that reducing the coupling strength could lead to a change from a first to second order percolation transition. In 2011, Huang et al. [17] mainly demonstrated the robustness of fully interdependent networks under targeted attack. The result implied that interdependent networks were difficult to defend. However, when real scenarios are considered, simultaneous attack on partially interdependent networks intentionally is more general.

Motivated by the above, we develop a generalized framework to comprehensively study the percolation of partially interdependent networks that suffer targeted attack simultaneously. Likewise, percolation law and condition of the first and the second phase transitions have been analyzed for partially interdependent networks. Furthermore, by applying generalized framework and analyzing the percolation, as α of $W_\alpha(k_i)$

varying, the corresponding percolation phase transition and critical line between the first and the second phase transition are firstly presented. Meanwhile, the robustness of two interdependent networks can be comprehensively studied by using the generalized framework.

II. THE MODEL

This model consists of two networks A, B with the number of nodes N_A, N_B , and within each network, the nodes are connected with degree distributions $P_A(k)$ and $P_B(k)$, respectively. Suppose that the average degree of the network A is a and the average degree of the network B is b . In addition, a fraction q_A of network A nodes depends on the nodes in network B and a fraction q_B of network B nodes depends on the nodes in network A . That is, if node A_i of network A depends on node B_j of network B and B_j depends on node A_k of network A , then $k = i$. Consequently, when nodes from one network fail, the interdependent nodes from the other network also fail. This invokes an iterative cascade of failures in both networks. A value $W_\alpha(k_i)$ is assigned to each node, which presents the probability that a node i with k_i links becomes inactive by targeted-attack. We focus on the family of functions [18]:

$$W_\alpha(k_i) = \frac{k_i^\alpha}{\sum_{i=1}^N k_i^\alpha}, -\infty < \alpha < +\infty. \quad (1)$$

When $\alpha \neq 0$, nodes are attacked intentionally, while for $\alpha = 0$, nodes are removed in random.

We begin by studying the situation where both networks A and B are attacked simultaneously with probability $W_\alpha(k_i)$ (Eq. (1)). Initially, $1 - p_1$ and $1 - p_2$ fraction of nodes are intentionally removed from network A and network B respectively. p_A and p_B are defined as the fraction of nodes belonging to the giant components of networks A and B . The remaining fraction of network A nodes after an initial removal

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of $1 - p_1$ is $\psi'_1 = p_1$, and the remaining fraction of network B nodes after an initial removal of $1 - p_2$ is $\phi'_0 = p_2$. The remaining functional part of network A contains a fraction $\psi_1 = \psi'_1 p_A(\psi'_1)$ of network nodes. Accordingly, the remaining fraction of network B is $\phi'_1 = p_2[1 - q_B(1 - p_A(\psi'_1)p_1)]$, and the fraction of nodes in the giant component of network B is $\phi_1 = \phi'_1 p_B(\phi'_1)$. Then the sequence, ψ_n and ϕ_n , of giant components, and the sequence ψ'_n and ϕ'_n , of the remaining fraction of nodes at each stage of the cascading failures, are constructed as follows:

$$\begin{aligned}
 \psi'_1 &= p_1, \psi_1 = \psi'_1 p_A(\psi'_1), \\
 \phi'_0 &= p_2, \phi'_1 = p_2[1 - q_B(1 - p_A(\psi'_1)p_1)], \phi_1 = \phi'_1 p_B(\phi'_1), \\
 \psi'_2 &= p_1[1 - q_A(1 - p_B(\phi'_1)p_2)], \psi_2 = \psi'_2 p_A(\psi'_2), \\
 \phi'_2 &= p_2[1 - q_B(1 - p_A(\psi'_2)p_1)], \phi_2 = \phi'_2 p_B(\phi'_2), \\
 &\dots \\
 \psi'_n &= p_1[1 - q_A(1 - p_B(\phi'_{n-1})p_2)], \psi_n = \psi'_n p_A(\psi'_n), \\
 \phi'_n &= p_2[1 - q_B(1 - p_A(\psi'_n)p_1)], \phi_n = \phi'_n p_B(\phi'_n).
 \end{aligned} \tag{2}$$

To determine the state of system (2) at the end of the cascading process we look at ψ'_n, ϕ'_n at the limit of $n \rightarrow \infty$. The limit must satisfy the equations $\psi'_n = \psi'_{n+1}, \phi'_n = \phi'_{n+1}$ since eventually the clusters stop fragmenting and the fractions of randomly removed nodes at step n and $n+1$ are equal. Denoting $\psi'_n = x, \phi'_n = y$, we arrive at a system of two symmetric equations:

$$\begin{aligned}
 x &= p_1[1 - q_A(1 - p_B(y)p_2)], \\
 y &= p_2[1 - q_B(1 - p_A(x)p_1)].
 \end{aligned} \tag{3}$$

For equation (1), as $\alpha = 0$, Fig. 1 show excellent agreement between computer simulations of the cascade failures and the numerical results obtained by solving system (2).

III. ANALYTICAL SOLUTION

Here we discuss the exact analytical results when $\alpha = 0$ and $\alpha = 1$. As in refs. [15, 16, 22, 23], we introduce the generating function of the degree distribution $G_{A0}(\xi) = \sum_k P_A(k)\xi^k$, and the generating function of the associated branching process, $G_{A1}(\xi) = G'_{A0}(\xi)/G'_{A0}(1)$. The fraction of nodes that belongs to the giant component after the removal of $1 - p_1$ nodes is [24]:

$$p_A(p_1) = 1 - G_{A0}[1 - p_1(1 - f_A)], \tag{4}$$

where $f_A = f_A(p_1)$ satisfies a transcendental equation

$$f_A = G_{A1}[1 - p_1(1 - f_A)]. \tag{5}$$

When $\alpha = 0$, $W_0 = \frac{1}{N}$, represents the random removal of nodes. For the case of two Erdős-Rényi (ER) [19–21] networks with average degrees a and b , we can easily get $p_A(x) = 1 - f_A, p_B(y) = 1 - f_B$, and system (3) becomes

$$\begin{aligned}
 x &= p_1[1 - q_A + p_2 q_A(1 - f_B)], \\
 y &= p_2[1 - q_B + p_1 q_B(1 - f_A)].
 \end{aligned} \tag{6}$$

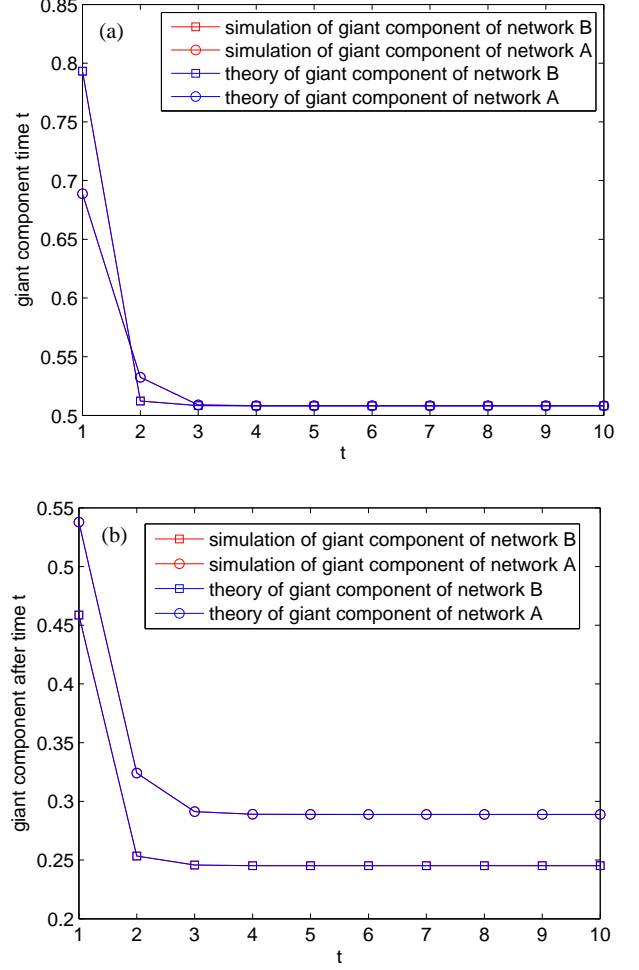


FIG. 1: (color online) (a) Simulation results of the giant component of the two fully interdependent ER networks after time t cascading failures. For each network $N = 100,000$, $a = b = 6$, $p_1 = 0.8$, $p_2 = 0.7$, $q_A = q_B = 1$. (b) Simulation results of the giant component of the two partially interdependent ER networks after t cascading failures. For each network $N = 100,000$, $a = b = 6$, $q_A = 0.6$, $q_B = 0.65$, $p_1 = 0.49$, $p_2 = 0.56$. All estimates are the results of averaging over 10 realizations. ψ_∞ and ϕ_∞ , the fraction of nodes in the giant components of networks A and B separately, after a cascade of failures, the simulations results for fully and partially interdependent networks are identical with the theoretical values.

The fraction of nodes in the giant components of networks A and B , at the end of the cascading process are given by:

$$\begin{aligned}
 \psi_\infty &= p_1(1 - f_A)[1 - q_A + p_2 q_A(1 - f_B)], \\
 \phi_\infty &= p_2(1 - f_B)[1 - q_B + p_1 q_B(1 - f_A)].
 \end{aligned} \tag{7}$$

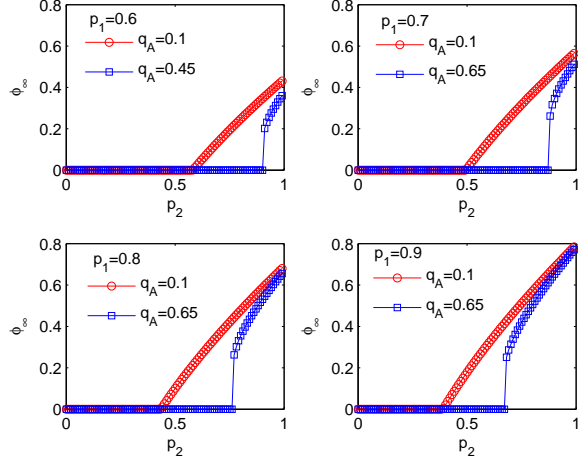


FIG. 2: (color online) For given $p_1 = 0.6 \sim 0.9$, with the strong coupling, the system undergoes the first phase transition at which ϕ_∞ , the fraction of nodes in the giant component of network B , abruptly jumps from a finite value to zero. However, with the weak coupling, the system undergoes the second phase transition at which ϕ_∞ , gradually approaches to zero.

And f_A, f_B can be expressed as:

$$\begin{aligned} f_A(f_B) &= \frac{1}{q_B} \left[\frac{1 + q_B(p_1 - 1)}{p_1} - \frac{\ln f_B}{bp_1 p_2 (f_B - 1)} \right], \\ f_A &\neq 1; \forall f_A, f_B = 1, \\ f_B(f_A) &= \frac{1}{q_A} \left[\frac{1 + q_A(p_2 - 1)}{p_2} - \frac{\ln f_A}{ap_1 p_2 (f_A - 1)} \right], \\ f_B &\neq 1; \forall f_A, f_B = 1. \end{aligned} \quad (8)$$

In fact, for random attack ($\alpha = 0$) on partially interdependent networks simultaneously, the change between the first phase transition and the second phase transition can be obtained by regulating the coupling strength q_A or q_B . Fig. 2 shows that reducing the coupling strength lead to a change from the first to second phase transition. And for different given p_1 , the critical line is also graphically founded ([Fig. 3]). From Fig. 3, we can observe that for $\alpha = 0$, the tendency of critical line is monotone decreasing with increasing parameters p_1 . And for $p_1 = 0.6, p_1 = 0.7, p_1 = 0.8$ and $p_1 = 0.9$, the corresponding percolation phase transitions are shown separately in Figure 3.

Especially when $a = b = k, p_1 = p_2 = p, q_A = q_B = q$, we arrive to the equations:

$$f_A = f_B = f = e^{kp(f-1)[1-q+pq(1-f)]}, 0 \leq f < 1, \quad (9)$$

and

$$\psi_\infty = \phi_\infty = p(1 - e^{-k\phi_\infty})[1 - q + pq(1 - e^{-k\phi_\infty})]. \quad (10)$$

Eq. (9) can be solved graphically as the intersection of a straight line $y = f$ and a curve $y = e^{kp(f-1)[1-q+pq(1-f)]}$. When p is small enough the curve increases very slowly and

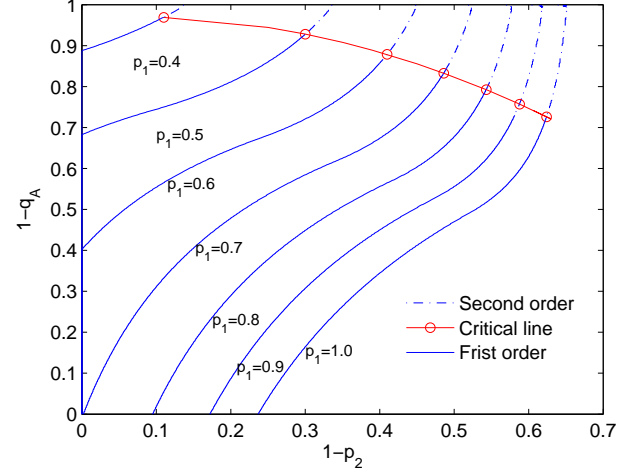


FIG. 3: (color online) The percolation phase transition for network B with $a = b = 3, q_B = 0.7$, and $p_1 = 0.4 \sim 1.0$. The critical line, as the boundary between the first and second phase transition, is founded. The tendency of critical line is monotone decreasing with p_1 . Below the critical line, the system undergoes a first order phase transition. Above the critical line, the system undergoes a second order transition.

does not intersect with the straight line except at $f = 1$ which corresponds to the trivial solution. The condition for the first order transition ($p = p^I$) is that the derivatives of the equations of system (9) with respect to f :

$$1 = f[kp^I(1 - q) + 2k(p^I)^2 q(1 - f)]. \quad (11)$$

And solving system (9) for $f \rightarrow 1$ yields the condition for the second order transition ($p = p^{II}$):

$$kp^{II}(1 - q) = 1. \quad (12)$$

The analysis of Eqs. (11) and (12) show that the first and second order transition lines coincide each other:

$$p = p^I = p^{II} = \frac{1}{k(1 - q)}. \quad (13)$$

The critical values of p_c, q_c for which the phase transition changes from the first order to second order are obtained when Eqs. (9), (11) and (12) are satisfied simultaneously, we get the critical values are only related with average degree as following:

$$\begin{aligned} p_c &= \frac{k + 1 - \sqrt{2k + 1}}{k}, \\ q_c &= \frac{\sqrt{2k + 1} + 1}{2k}. \end{aligned} \quad (14)$$

The phase transition and critical line are graphically showed respectively in Fig. 4. From Fig. 4, the tendency of critical line descend with average degree k .

When $\alpha \neq 0$, both of the two partially interdependent networks are initially attacked with probability $W_\alpha(k_i)$ (Eq. (1)).

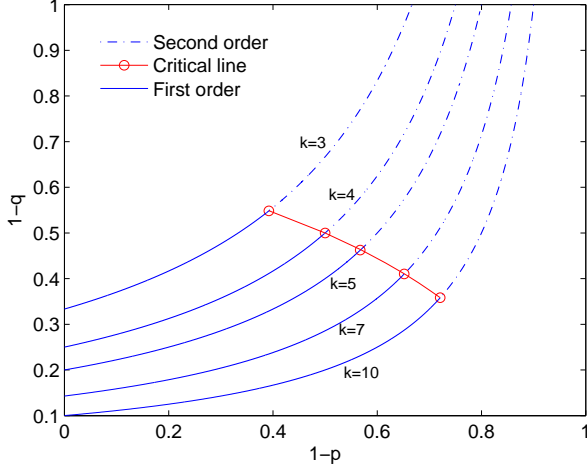


FIG. 4: (color online) The phase transition is showed for $k = 3, 4, 5, 7, 10$. The critical line is also graphically presented by Eq. (14). Its tendency is monotone decreasing with increasing average k .

Initially, $1 - p_1$ and $1 - p_2$ fraction of nodes are intentionally removed from network A and network B respectively. The generating functions will be defined for network A while similar equations describe network B . According to Eq. (1), $1 - p_1$ fraction of nodes are removed from network A but before the links of the remaining nodes which connect to the removed nodes are removed. The generating function of the nodes left in network A before removing the links to the removed nodes is [17, 23, 25]

$$G_{Ab}(x) = \sum_{k_1} P_{p_1}(k_1) x^{k_1} = \frac{1}{p_1} \sum_{k_1} P(k_1) h_1^{k_1^\alpha} x^{k_1}, \quad (15)$$

where the new degree distribution of the remaining fraction p_1 of nodes $P_{p_1}(k_1) = \frac{1}{p_1} P(k_1) h_1^{k_1^\alpha}$, and $G_\alpha(x) \equiv \sum_{k_1} P(k_1) x^{k_1^\alpha}$, $h_1 \equiv G_\alpha^{-1}(p_1)$. The generating function of the new distribution of nodes left in network A after the links to the removed nodes are also removed is

$$G_{Ac}(x) = G_{Ab}(1 - \tilde{p}_1 + \tilde{p}_1 x), \quad (16)$$

where $\tilde{p}_1 = \frac{p_1 N_A \langle k_1(p_1) \rangle}{N_A \langle k_1 \rangle} = \frac{\sum_{k_1} P(k_1) k_1 h_1^{k_1^\alpha}}{\sum_{k_1} P(k_1) k_1}$ is the fraction of the original links that connect to the nodes left, $\langle k_1 \rangle$ is the average degree of the original network A , $\langle k_1(p_1) \rangle$ is the average degree of remaining nodes before the links that are disconnected are removed. Then we can find equivalent networks A' and B' with generating functions $\tilde{G}_{A0}(x)$ and $\tilde{G}_{B0}(x)$, such that after simultaneous random attack with $1 - p_1$ and $1 - p_2$ fractions of nodes, the new generating functions of nodes left in A' and B' are the same as $G_{Ac}(x)$ and $G_{Bc}(x)$. That is, the simultaneous targeted-attack problem on interdependent networks A and B can be solved as simultaneous random-attack problem on interdependent networks A' and B' . By solving the equations $\tilde{G}_{A0}(1 - p_1 + p_1 x) = G_{Ac}(x)$,

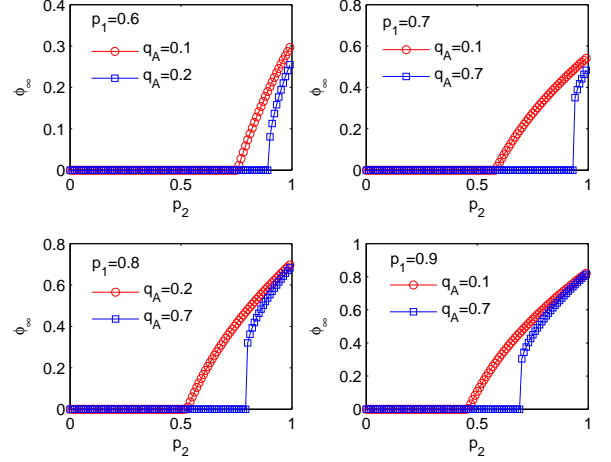


FIG. 5: (color online) For given $p_1 = 0.6 \sim 0.9$, the system undergoes the first phase transition and the second phase transition for network B with strong and weak coupling.

$\tilde{G}_{B0}(1 - p_2 + p_2 x) = G_{Bc}(x)$ and from Eq. (16), we can get

$$\begin{aligned} \tilde{G}_{A0}(x) &= G_{Ab}(1 - \frac{\tilde{p}_1}{p_1} + \frac{\tilde{p}_1}{p_1} x), \\ \tilde{G}_{B0}(x) &= G_{Bb}(1 - \frac{\tilde{p}_2}{p_2} + \frac{\tilde{p}_2}{p_2} x). \end{aligned} \quad (17)$$

Simplified forms for $G_{Ab}(x)$, $G_{Ac}(x)$ and $\tilde{G}_{A0}(x)$ from Eqs. (15), (16) and (17) exist when $\alpha = 1$,

$$G_{Ab}(x) = \frac{1}{p_1} \sum_{k_1} P(k_1) h_1^{k_1} x^{k_1} = \frac{1}{p_1} G_{A0}(h_1 x), \quad (18)$$

$$G_{Ac}(x) = \frac{1}{p_1} G_{A0}(h_1(1 - \tilde{p}_1 + \tilde{p}_1 x)), \quad (19)$$

$$\tilde{G}_{A0}(x) = \frac{1}{p_1} G_{A0}(h_1(1 - \frac{\tilde{p}_1}{p_1} + \frac{\tilde{p}_1}{p_1} x)), \quad (20)$$

where $G_{A0}(x)$ is the original generating function of the network A , $h_1 = G_{A0}^{-1}(p_1)$, $\tilde{p}_1 = \frac{G'_{A0}(h_1)}{G_{A0}(1)} h_1$. For ER [19–21] networks, we can also get $p_A(x) = 1 - f_A$, $p_B(y) = 1 - f_B$, where $p_A(x) = 1 - \tilde{G}_{A0}[1 - x(1 - f_A)]$, $f_A = \tilde{G}_{A1}[1 - x(1 - f_A)]$, and system (3) becomes

$$\begin{aligned} x &= p_1[1 - q_A + p_2 q_A(1 - f_B)], \\ y &= p_2[1 - q_B + p_1 q_B(1 - f_A)]. \end{aligned} \quad (21)$$

The fraction of nodes in the giant components of networks A and B , respectively, at the end of the cascade process are then given by:

$$\begin{aligned} \psi_\infty &= p_1(1 - f_A)[1 - q_A + p_2 q_A(1 - f_B)], \\ \phi_\infty &= p_2(1 - f_B)[1 - q_B + p_1 q_B(1 - f_A)]. \end{aligned} \quad (22)$$

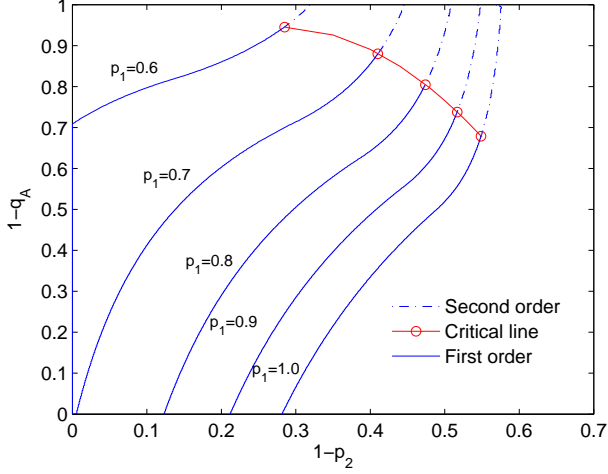


FIG. 6: (color online) The percolation phase transition for network B with $a = 3$, $b = 4$, $q_B = 0.7$ and $p_1 = 0.6 \sim 1.0$. The corresponding phase transition lines are showed. Also, the critical line is graphically found, and its tendency is monotone decreasing with p_1 . Below the critical line, the system undergoes a first order phase transition. As we approach the critical point, ϕ_∞ tends to 0. Above the critical line, the system undergoes a second order transition.

And f_A , f_B can be expressed as:

$$f_A(f_B) = \frac{1}{q_B} \left[\frac{1 + q_B(p_1 - 1)}{p_1} - \frac{\ln f_B}{b p_1 p_2 h_2^2 (f_B - 1)} \right],$$

$$f_A \neq 1; \forall f_A, f_B = 1,$$

$$f_B(f_A) = \frac{1}{q_A} \left[\frac{1 + q_A(p_2 - 1)}{p_2} - \frac{\ln f_A}{a p_1 p_2 h_1^2 (f_A - 1)} \right],$$

$$f_B \neq 1; \forall f_A, f_B = 1. \quad (23)$$

Likewise, for targeted attack ($\alpha = 1$) on partially interdependent networks simultaneously, the phase transition changes from a first order to a second order percolation transition, as the coupling strength q_A is reduced (Fig. 5). From Fig. 6, the phase transition lines are graphically presented, and the critical line descends with increasing parameters p_1 .

Especially as $a = b = k$, $p_1 = p_2 = p$, $q_A = q_B = q$, we obtain the following equations:

$$f_A = f_B = f = e^{-k p h^2 (1-f)[1-q+p q(1-f)]}, \quad (24)$$

and

$$\psi_\infty = \phi_\infty = p(1 - e^{-k h^2 \phi_\infty})[1 - q + p q(1 - e^{-k h^2 \phi_\infty})], \quad (25)$$

where $h = \frac{\ln p}{k} + 1$. The condition for the first order transition ($p = p^I$) is that the derivatives of system (24) with respect to f :

$$1 = f[k p^I h^2 (1 - q) + 2k(p^I)^2 q h^2 (1 - f)], 0 \leq f < 1. \quad (26)$$

And solving system (24) for $f \rightarrow 1$ yields the condition for the second order transition ($p = p^{II}$):

$$k p^{II} (1 - q) h^2 = 1. \quad (27)$$

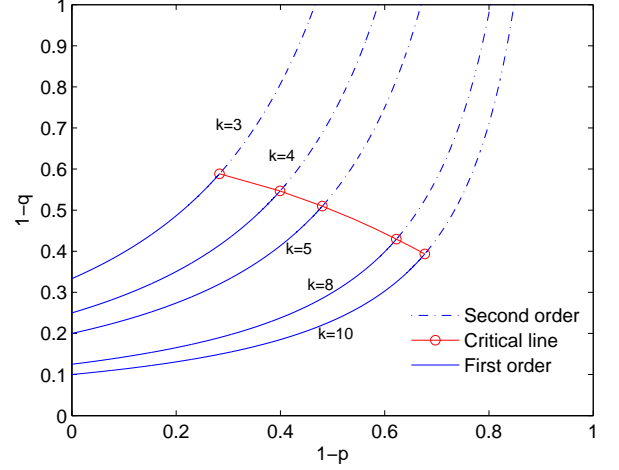


FIG. 7: (color online) The corresponding phase transition is showed for $k = 3, 4, 5, 8, 10$. The critical line is also graphically presented from simulation. The tendency of critical line is found monotone decreasing with average degree k .

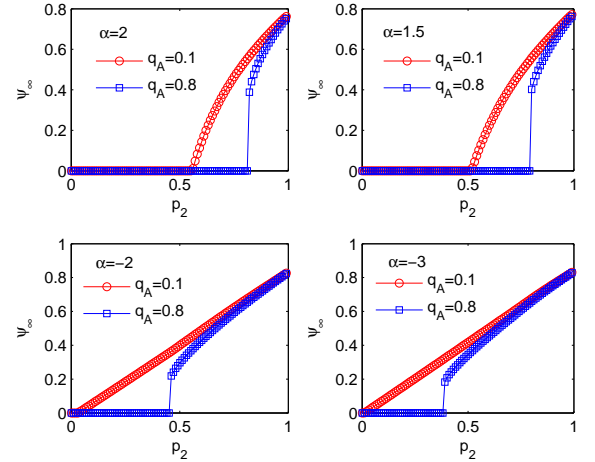


FIG. 8: (color online) For given $\alpha = 2, 1.5, -2, -3$, the system undergoes the first phase transition and the second phase transition for network A with strong and weak coupling.

From Eqs. (26) and (27), we have also found that the first order transition line coincides with the second order transition line ($p = p^I = p^{II}$), the exact condition of phase transition is as follows:

$$p \left(\frac{\ln p}{k} + 1 \right)^2 = \frac{1}{k(1 - q)}. \quad (28)$$

From Fig. 7, the simulation of phase transition is showed for Eq. (28). Meanwhile, the critical line is also by simulation. From Fig. 7, the tendency of critical line is monotone decreasing with average degree k .

Both networks A and B are attacked simultaneously with probability $W_\alpha(k_i)$, when $\alpha > 0$, nodes with higher degree are more vulnerable and those nodes are intentionally

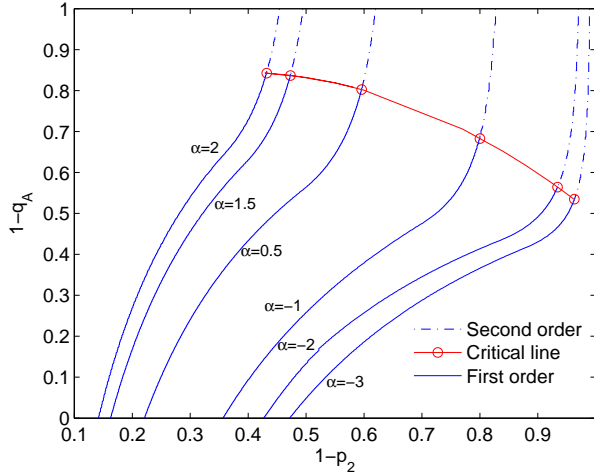


FIG. 9: (color online) The percolation phase transition for network A with $a = 3$, $b = 4$, $q_B = 0.7$, $p_1 = 0.8$ and $\alpha = -3, -2, -1, 0.5, 1.5, 2$. The corresponding phase transitions lines are showed. Also, the critical line is graphically found, and its tendency is monotone increasing with α . Below the critical line, the system undergoes a first order phase transition. As we approach the critical point, ψ_∞ tends to 0. Above the critical line, the system undergoes a second order transition.

attacked, when $\alpha < 0$ node with higher degree have lower probability to fail. For different values of α , Fig. 8 reflect the relationship between the phase transition and the coupling strength. Fig. 9 show the phase transition lines for different given $\alpha > 0$ respectively. The critical line is also founded and its tendency is monotone increasing with α .

IV. CONCLUSIONS

For interdependent networks in real scenario, two factors are necessary to be considered: partial coupling and targeted attack. A general framework is proposed to investigate the percolation of partially interdependent networks that suffer targeted-attack simultaneously. The percolation of partially interdependent networks under targeted attack is comprehensively analyzed. As $\alpha = 0$ and $\alpha = 1$, the percolation law is described detailedly. Especially, for $a = b = k$, $p_1 = p_2 = p$, $q_A = q_B = q$, the first and second lines of phase transition coincide with each other. And, the tendency of critical line monotone decreasing with average degree k . we show both analytically and numerically that reducing the coupling between the networks leads to a change from a first to a second order phase transition at a critical line. The tendency of critical line is monotone decreasing with parameter p_1 . However, for different α , the percolation phase transition is also graphically demonstrated and critical line is monotone increasing with α . Therefore, our finding should be worth considering in designing robust network considering.

Acknowledgments

This work is funded by the National Natural Science Foundation of China (Grant Nos. 91010011, 71073072, 51007032), the Natural Science Foundation of Jiangsu Province (Grant No. 2007098), the National Natural Science (Youth) Foundation of China (Grant No. 10801140), the Graduate innovative Foundation of Jiangsu Province CX10B_272Z and the Youth Foundation of Chongqing Normal University (Grant No. 10XLQ001).

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